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Radiation reaction and 4-momentum conservation for point-like dyons

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Abstract
We construct for a system of point-like dyons a conserved energy–momentum tensor entailing finite momentum integrals that takes the radiation reaction into account.

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1. Introduction

A charged point particle creates an electromagnetic field strength, the Lienard–Wiechert field, that diverges at the position of the particle. This implies that the radiation reaction, or self-force, experienced by the particle is infinite, and the Lorentz equation is ill defined. The divergent contribution from this equation can be eliminated by a classical infinite mass renormalization, to obtain a finite effective equation of motion, the Lorentz–Dirac equation [1], that plays a crucial role in classical radiation theory. This equation takes the radiation reaction into account and completes the Larmor formula.

The Lorentz–Dirac effective equation of motion bears several unusual features. It is of third order in the time derivative, and it cannot be deduced from a standard Lagrangian: eventually it must be postulated. The ultimate justification for the equation comes from the requirement of *local energy–momentum conservation*, i.e. there should exist an energy–momentum tensor $T^{\mu\nu}$ that is (a) conserved, and (b) admits finite momentum integrals. The *naive* energy–momentum tensor $\Theta^{\mu\nu}$, being quadratic in the field strength $F^{\mu\nu}$, diverges at the position of the particle-like $\Theta^{\mu\nu} \sim 1/R^4$, if R is the distance from the particle, and it does therefore not admit finite 4-momentum integrals. Said differently, while $F^{\mu\nu}$ is a distribution $\Theta^{\mu\nu}$ is not a distribution, because the square of a distribution is in general not a distribution. Consequently, the 4-divergence $\partial_\mu \Theta^{\mu\nu}$ does not even make sense.

The construction of a consistent energy–momentum tensor requires, in some sense, to isolate and subtract from $\Theta^{\mu\nu}$ the singularity present at the position of the particle, *without*

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modifying the value of $\Theta^{\mu\nu}$ *in the complement of the particle's worldline*, in compatibility with energy–momentum conservation and Lorentz invariance. More precisely, the so-obtained renormalized energy–momentum tensor should be conserved, *if the particle satisfies the Lorentz–Dirac equation of motion*. It is clear that such a program can be carried out only in the framework of distribution theory.

Until now the construction outlined here has been realized only for a charged point particle in four dimensions, rather recently [2], using a somewhat cumbersome and implicit distribution technique. A physically more transparent alternative representation of the resulting energy–momentum tensor—again in the framework of distribution theory—has been given in [3], relying on a new Lorentz-invariant regularization scheme, followed by a classical renormalization.

Aim of the present paper is to generalize the new approach of [3] to construct a consistent energy–momentum tensor for a system of point-like dyons, taking the radiation reaction into account, which has not been given before[1]. This result completes the consistency proof for a classical system of radiating dyons, satisfying generalized duality invariant Lorentz–Dirac equations [4, 5], see (2.8).

The new method illustrated here, due to its manifest Lorentz invariance at each step, appears in particular suitable for extension to a system of strings or branes in arbitrary dimensions. Using this method we hope indeed to furnish elsewhere the construction of a consistent energy–momentum tensor for a generic radiating extended object that is still unknown.

2. Regularized Maxwell equations

For simplicity we consider a single dyon with mass m, electric and magnetic charges e and g, and worldline $y^\mu(s)$, the extension of our construction to a system of N dyons being straightforward. We denote 4-velocity, 4-acceleration and derivative of the 4-acceleration by $u^\mu = \mathrm{d}y^\mu/\mathrm{d}s$, $w^\mu = \mathrm{d}u^\mu/\mathrm{d}s$, $b^\mu = \mathrm{d}w^\mu/\mathrm{d}s$. Introducing a current with unit charge as

$$j^\mu(x) = \int u^\mu \delta^4(x-y)\,\mathrm{d}s, \tag{2.1}$$

the electric and magnetic currents are $j_e^\mu = ej^\mu$ and $j_m^\mu = gj^\mu$. The Maxwell equations for the dyon become then

$$\partial_\mu F^{\mu\nu} = j_e^\nu, \tag{2.2}$$

$$\partial_\mu \widetilde{F}^{\mu\nu} = j_m^\nu, \tag{2.3}$$

where we indicate the dual of an antisymmetric tensor with $\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. The general solution of (2.2), (2.3) can be written as

$$F^{\mu\nu} = f^{\mu\nu} + eH^{\mu\nu} - g\widetilde{H}^{\mu\nu}, \qquad H^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.4}$$

where $f^{\mu\nu}$ is a free radiation field, $\partial_\mu f^{\mu\nu} = 0 = \partial_\mu \widetilde{f}^{\mu\nu}$, and A^μ is a unit Lienard–Wiechert potential in the Lorentz gauge, satisfying $\Box A^\mu = j^\mu$, $\partial_\mu A^\mu = 0$,

$$A^\mu = \frac{u^\mu}{4\pi(uR)}. \tag{2.5}$$

We write the scalar products as $a^\mu b_\mu = (ab)$, and we have defined

$$R^\mu(x) \equiv x^\mu - y^\mu(s). \tag{2.6}$$

[1] The energy–momentum tensor proposed in [4] requires to modify the naive tensor $\Theta^{\mu\nu}$ *also* in the complement of the worldline.

The proper time appearing in y^μ and in u^μ is the retarded proper time $s(x)$ determined from

$$(x - y(s))^2 = 0, \quad x^0 > y^0(s). \tag{2.7}$$

This means in particular that we have $R^\mu R_\mu = 0$, and hence $R^0 = |\vec{R}| \equiv R$.

Since A^μ carries $1/R$ singularities, the unit field strength $H^{\mu\nu}$ carries $1/R^2$ singularities near the worldline, and the Lorentz equation for the dyon would be singular. After subtraction of the singularity one postulates the following finite duality invariant generalization of the Lorentz–Dirac equation for a dyon [5], $p^\mu = mu^\mu$, $w^2 = w^\mu w_\mu$,

$$\frac{\mathrm{d}p^\mu}{\mathrm{d}s} = \frac{e^2 + g^2}{6\pi}\left(\frac{\mathrm{d}w^\mu}{\mathrm{d}s} + w^2 u^\mu\right) + (ef^{\mu\nu} + g\widetilde{f}^{\mu\nu})u_\nu \tag{2.8}$$

that takes the radiation reaction into account. For $g = 0$ one gets back the Lorentz–Dirac equation.

On the other hand, while A and F have at most $1/R^2$ three-space integrable singularities near the worldline of the particle and are distributions, the naive electromagnetic energy–momentum tensor[2],

$$\Theta^{\mu\nu} = (FF)^{\mu\nu} - \tfrac{1}{4}\eta^{\mu\nu}(FF), \tag{2.9}$$

carries three-space *non-integrable* $1/R^4$ singularities, and is not a distribution.

As a first step to isolate the singularities in $\Theta^{\mu\nu}$ we introduce a Lorentz-invariant regularization, parametrized by a positive regulator with the dimension of length ε, by replacing the retarded proper time $s(x)$ appearing in A^μ in (2.5), with a regularized retarded proper time $s_\varepsilon(x)$, determined from

$$(x - y(s))^2 = \varepsilon^2, \qquad x^0 > y^0(s). \tag{2.10}$$

We call the resulting regularized potential

$$A^\mu_\varepsilon = \left.\frac{u^\mu}{4\pi(uR)}\right|_{s=s_\varepsilon(x)}, \qquad \partial_\mu A^\mu_\varepsilon = 0, \tag{2.11}$$

where from now on with y^μ, u^μ, R^μ etc we intend their regularized versions, obtained through the replacement $s(x) \to s_\varepsilon(x)$. The regularized field strength becomes then

$$F^{\mu\nu}_\varepsilon = f^{\mu\nu} + eH^{\mu\nu}_\varepsilon - g\widetilde{H}^{\mu\nu}_\varepsilon, \tag{2.12}$$

where

$$H^{\mu\nu}_\varepsilon = \partial^\mu A^\nu_\varepsilon - \partial^\nu A^\mu_\varepsilon. \tag{2.13}$$

We define the regularized energy–momentum tensor as

$$\Theta^{\mu\nu}_\varepsilon = (F_\varepsilon F_\varepsilon)^{\mu\nu} - \tfrac{1}{4}\eta^{\mu\nu}(F_\varepsilon F_\varepsilon). \tag{2.14}$$

The fields A_ε, F_ε and Θ_ε are now all regular on the particle's worldline, indeed they are C^∞ functions on $\mathbf{R}^4$. But, whereas A_ε and F_ε for $\varepsilon \to 0$ tend to A and F in the distributional sense, Θ_ε converges to Θ *pointwise away from the worldline*, but not in the distributional sense, because Θ is not a distribution[3].

[2] In the following we use for the contraction of two antisymmetric tensors the notation $(AB)^{\mu\nu} = A^\mu{}_\rho B^{\rho\nu}$ and $(AB) = A^{\mu\nu}B_{\nu\mu}$.

[3] Saying that a set of functions, or distributions, f_ε converges for $\varepsilon \to 0$ to f in the distributional sense means that it converges if applied to an arbitrary test function, i.e. one has $\lim_{\varepsilon\to 0} f_\varepsilon(\varphi) = f(\varphi)$ for every $\varphi \in \mathcal{S}(\mathbf{R}^4)$.

3. Construction of a finite energy–momentum tensor

Before taking the distributional limit of Θ_ε we must therefore separate and subtract its contributions that diverge for $\varepsilon \to 0$ in the distributional sense. This means that we have to apply Θ_ε to a generic test function φ, and isolate the terms of the resulting integral that diverge for $\varepsilon \to 0$. The rest of this section is mainly devoted to the explicit identification, and subtraction, of these singular terms, the result being formula (3.5).

To begin with we need an explicit expression for F_ε, i.e. for H_ε. Differentiating (2.10) to derive $\partial_\mu s_\varepsilon = R_\mu/(uR)$, from (2.11) one obtains

$$\partial^\mu A^\nu_\varepsilon = \frac{1}{4\pi(uR)^3}[(\eta^{\mu\rho} - u^\mu u^\rho)R_\rho u^\nu + (u^\rho w^\nu - u^\nu w^\rho)R_\rho R^\mu]. \tag{3.1}$$

The most singular terms in H_ε, see (2.13), go therefore as $1/R^2$ near the worldline, and since the radiation field $f^{\mu\nu}$ is supposed to be regular, the contributions of Θ_ε that do not converge for $\varepsilon \to 0$ in the distributional sense, are only those quadratic in H_ε. Inserting (2.12) in (2.14) and keeping only the terms quadratic in H_ε, one obtains for the divergent part of Θ_ε therefore,

$$\Theta^{\mu\nu}_\varepsilon\big|_{\text{div}} = (e^2 + g^2)\big[(H_\varepsilon H_\varepsilon)^{\mu\nu} - \tfrac{1}{4}\eta^{\mu\nu}(H_\varepsilon H_\varepsilon)\big]_{\text{div}}. \tag{3.2}$$

Note that the cross terms in e and g get cancelled.

We remain then with the evaluation of $(H_\varepsilon H_\varepsilon)_{\text{div}}$. This product contains terms that behave as $1/R^n$ near the worldline, with $n = 2, 34$. As $\varepsilon \to 0$ in the distributional sense, for dimensional reasons the terms with $n = 4$ give rise to simple pole [$\sim 1/\varepsilon$], and logarithmic [$\sim \ln\varepsilon$] singularities, those with $n = 3$ give rise to logarithmic singularities, while those with $n = 2$ are convergent. Actually, it can be seen that the logarithmic singularities cancel between the $n = 4$ and $n = 3$ terms, and one remains only with the pole singularities contained in the $1/R^4$ terms. To determine $(H_\varepsilon H_\varepsilon)_{\text{div}}$ it is then sufficient to evaluate the simple pole term of, see (3.1),

$$\left(\partial^\mu A^\nu_\varepsilon \partial^\alpha A^\beta_\varepsilon\right)_{\text{div}} = \frac{1}{16\pi^2}(\eta^{\mu\rho} - u^\mu u^\rho)(\eta^{\alpha\sigma} - u^\alpha u^\sigma)u^\nu u^\beta \left.\frac{R_\rho R_\sigma}{(uR)^6}\right|_{1/\varepsilon}, \tag{3.3}$$

where we kept only the $1/R^4$ terms. The rest of this section is devoted to the explicit evaluation of the rhs of this formula.

Since the divergences as $\varepsilon \to 0$ are intended in the distributional sense, we must apply the rhs of (3.3) to a test function. Omitting for simplicity of writing the (regular) tensorial prefactor $(\eta^{\mu\rho} - u^\mu u^\rho)\cdots$ in (3.3), we have to evaluate the function $R_\rho R_\sigma/(uR)^6$ applied to a generic test function $\varphi(x) \in \mathcal{S}$, i.e. by definition,

$$\frac{R_\rho R_\sigma}{(uR)^6}(\varphi) \equiv \int \frac{d^4x}{(uR)^6} R_\rho R_\sigma \varphi(x) = \int ds \int \frac{d^4x}{(ux)^5} 2\delta(x^2 - \varepsilon^2)x_\rho x_\sigma \varphi(x + y), \tag{3.4}$$

where we have inserted a δ-function to take the constraint (2.10) into account, and we have performed the shift $x^\mu \to x^\mu + y^\mu$. The x-integration is restricted to $x^0 > 0$. The kinematical quantities y and u are now evaluated at s, that is a free integration variable. The pole singularity of this expression can be extracted by rescaling $x^\mu \to \varepsilon x^\mu$, and sending in the integral ε to zero[4],

$$\left.\frac{R_\rho R_\sigma}{(uR)^6}(\varphi)\right|_{1/\varepsilon} = \frac{1}{\varepsilon}\int ds \int \frac{d^4x}{(ux)^5} 2\delta(x^2 - 1)x_\rho x_\sigma \varphi(y) = \frac{\pi^2}{4\varepsilon}\int ds(5u_\rho u_\sigma - \eta_{\rho\sigma})\varphi(y).$$

[4] The integral (3.4) contains actually also logarithmic divergences $\sim \ln\varepsilon$, but these can be seen to cancel against the logarithmic divergences present in the $1/R^3$ terms, as mentioned above.

This means that, as expected, the part that diverges as $\varepsilon \to 0$ in the distributional sense, is entirely supported on the worldline. Indeed, we can write the above result as

$$\left.\frac{R_\rho R_\sigma}{(uR)^6}\right|_{1/\varepsilon} = \frac{\pi^2}{4\varepsilon} \int (5u_\rho u_\sigma - \eta_{\rho\sigma})\delta^4(x-y)\,\mathrm{d}s.$$

Inserting this expression in (3.3) we obtain

$$\left(\partial^\mu A^\nu_\varepsilon \partial^\alpha A^\beta_\varepsilon\right)_{\mathrm{div}} = \frac{1}{64\varepsilon} \int (u^\alpha u^\mu - \eta^{\alpha\mu}) u^\nu u^\beta \delta^4(x-y)\,\mathrm{d}s.$$

Using this result in (3.2) allows one to determine the divergent part of the energy–momentum tensor as

$$\left.\Theta^{\mu\nu}_\varepsilon\right|_{\mathrm{div}} = \frac{e^2+g^2}{32\varepsilon} \int \left(u^\mu u^\nu - \frac{1}{4}\eta^{\mu\nu}\right)\delta^4(x-y)\,\mathrm{d}s.$$

This means that we can define a 'renormalized' energy–momentum tensor for the electromagnetic field as

$$T^{\mu\nu}_{\mathrm{em}} = \mathcal{S}' - \lim_{\varepsilon\to 0}\left[\Theta^{\mu\nu}_\varepsilon - \frac{e^2+g^2}{32\varepsilon}\int \left(u^\mu u^\nu - \frac{1}{4}\eta^{\mu\nu}\right)\delta^4(x-y)\,\mathrm{d}s\right], \tag{3.5}$$

where $\mathcal{S}' - \lim$ means a limit in the distributional sense. What we have shown here is that this limit exists and represents a well-defined distribution. This means in particular that the 4-momentum integrals over an arbitrary finite 3-volume $\int_V \mathrm{d}^3x\, T^{\mu 0}_{\mathrm{em}}$ exist, whether or not the particle at the given instant is inside V. If the acceleration of the particle vanishes sufficiently fast for $t \to -\infty$, then also the total 4-momentum is finite [2], see equation (4.9) for an explicit expression. Note also that the expression for $T^{\mu\nu}_{\mathrm{em}}$, apart from being manifestly Lorentz invariant, coincides with the naive tensor $\Theta^{\mu\nu}$ in the complement of the worldline. This feature realizes the requirement that the energy–momentum tensor should be 'changed only at the position of the particle'. The counterterm $u^\mu u^\nu$ in (3.5) can be interpreted as a kind of mass term, while the term proportional to $\eta^{\mu\nu}$ is needed to keep $T^{\mu\nu}_{\mathrm{em}}$ traceless.

4. Energy–momentum conservation

Above we have constructed an energy–momentum tensor that gives rise to finite momentum integrals. Given the (*a priori*) arbitrariness of our construction, its physical justification arises from the fulfilment of local energy–momentum conservation. The check of this conservation law requires the evaluation of the 4-divergence $\partial_\mu T^{\mu\nu}_{\mathrm{em}}$. The present section is devoted to this evaluation, the result being given in (4.8).

We begin by stating the form the (regularized) Maxwell equations satisfied by $F^{\mu\nu}_\varepsilon$. From (2.12) and (2.13) one obtains

$$\partial_\mu F^{\mu\nu}_\varepsilon = e j^\nu_\varepsilon, \qquad \partial_\mu \widetilde{F}^{\mu\nu}_\varepsilon = g j^\nu_\varepsilon, \tag{4.1}$$

where the regularized unit current,

$$j^\mu_\varepsilon \equiv \Box A^\mu_\varepsilon, \tag{4.2}$$

which is still conserved, can be calculated by applying one more derivative to (3.1),

$$j^\mu_\varepsilon = \frac{\varepsilon^2}{4\pi}\left(\frac{1}{(uR)^4}[(uR)b^\mu - (bR)u^\mu] + 3(1-(wR))^2\frac{u^\mu}{(uR)^5} + 3(1-(wR))\frac{w^\mu}{(uR)^4}\right). \tag{4.3}$$

The factor of ε^2 arises from the fact that $R^\mu R_\mu = \varepsilon^2$ and it implies that for $\varepsilon \to 0$, in the complement of the wordline j_ε^μ converges pointwise to zero. More precisely, one has the distributional limit $\mathcal{S}' - \lim_{\varepsilon\to 0} j_\varepsilon^\mu = j^\mu$, as implied by (4.2).

We now come back to the evaluation of $\partial_\mu T_{\rm em}^{\mu\nu}$. Since the convergence in (3.5) is in the distributional sense and since the distributional derivative is a continuous operation, we can interchange the derivative with the limit

$$\partial_\mu T_{\rm em}^{\mu\nu} = \mathcal{S}' - \lim_{\varepsilon\to 0}\left(\partial_\mu \Theta_\varepsilon^{\mu\nu} - \frac{e^2+g^2}{32\varepsilon}\int\left(w^\nu - \frac{1}{4}\partial^\nu\right)\delta^4(x-y)\,{\rm d}s\right). \tag{4.4}$$

Using the regularized Maxwell equations (4.1), one obtains

$$\partial_\mu \Theta_\varepsilon^{\mu\nu} = -\big(eF_\varepsilon^{\nu\mu} + g\widetilde{F}_\varepsilon^{\nu\mu}\big)j_{\varepsilon\mu} \tag{4.5}$$

$$= -(e^2+g^2)H_\varepsilon^{\nu\mu}j_{\varepsilon\mu} - (ef^{\nu\mu} + g\widetilde{f}^{\nu\mu})j_{\varepsilon\mu}. \tag{4.6}$$

In the terms containing the external field one can simply take the limit $j_{\varepsilon\mu} \to j_\mu$, whereas in the first term one has to evaluate carefully the distributional limit of the product $H_\varepsilon^{\nu\mu}j_{\varepsilon\mu}$. Due to the factor ε^2 in front of (4.3), this product converges pointwise to zero in the complement of the worldline; this means that its distributional limit—if it exists—is necessarily supported on the worldline. On the other hand, the distributional limit $\mathcal{S}' - \lim_{\varepsilon\to 0} H_\varepsilon^{\nu\mu}j_{\varepsilon\mu}$ cannot exist. Indeed, since the 4-divergence of a distribution is again a distribution, we know that the limit in (4.4) exists, and this implies that the divergent contributions of $H_\varepsilon^{\nu\mu}j_{\varepsilon\mu}$ must precisely compensate the $1/\varepsilon$ terms in (4.4). Thus, actually it is sufficient to determine the finite contributions of $H_\varepsilon^{\nu\mu}j_{\varepsilon\mu}$, as $\varepsilon \to 0$ in the distributional sense.

Using the techniques illustrated in the appendix it can indeed be shown that one has

$$H_\varepsilon^{\nu\mu}j_{\varepsilon\mu} = \int\left[\frac{1}{6\pi}\left(\frac{{\rm d}w^\nu}{{\rm d}s} + w^2u^\nu\right) - \frac{1}{32\varepsilon}\left(w^\nu - \frac{1}{4}\partial^\nu\right)\right]\delta^4(x-y)\,{\rm d}s + o(\varepsilon), \tag{4.7}$$

where $o(\varepsilon)$ stands for terms that go to zero as $\varepsilon \to 0$ in the distributional sense. Using this in (4.6) one sees that the $1/\varepsilon$ terms in (4.4) cancel, and one obtains

$$\partial_\mu T_{\rm em}^{\mu\nu} = -\int\left[\frac{e^2+g^2}{6\pi}\left(\frac{{\rm d}w^\nu}{{\rm d}s} + w^2u^\nu\right) + (ef^{\nu\mu} + g\widetilde{f}^{\nu\mu})u_\mu\right]\delta^4(x-y)\,{\rm d}s. \tag{4.8}$$

This implies eventually that, when adding the energy–momentum tensor of the dyon, $T^{\mu\nu} = T_{\rm em}^{\mu\nu} + m\int u^\mu u^\nu\delta^4(x-y)\,{\rm d}s$, the total energy–momentum tensor is conserved, if the generalized Lorentz–Dirac equation (2.8) holds.

If the external field vanishes, $f^{\mu\nu} = 0$, equation (4.8) can be integrated over whole three space at fixed time t, to obtain the derivative of the total 4-momentum of the electromagnetic field, $\frac{{\rm d}P_{\rm em}^\mu}{{\rm d}t}$. If the 4-acceleration of the dyon vanishes sufficiently fast for $t \to -\infty$, a further integration gives then the total 4-momentum of the electromagnetic field as

$$P_{\rm em}^\mu(t) = \int {\rm d}^3x\, T_{\rm em}^{0\mu} = -\frac{e^2+g^2}{6\pi}\left(w^\mu(s) + \int_{-\infty}^{s} w^2(\lambda)u^\mu(\lambda)\,{\rm d}\lambda\right), \tag{4.9}$$

where s is the proper time of the particle at the instant t.

5. Interpretation and outlook

The construction of the energy–momentum tensor performed here supplies further evidence of the consistency of the classical dynamics of a radiating dyon system. Indeed, for a system of particles our construction generalizes simply by replacing in (3.5) the counterterm with

$\frac{1}{32\varepsilon}\sum_r\left(e_r^2+g_r^2\right)\int\left(u_r^\mu u_r^\nu-\frac{1}{4}\eta^{\mu\nu}\right)\delta^4(x-y_r)\,\mathrm{d}s_r$. This subtraction is sufficient since the mutual interactions do not give rise to singularities in $\Theta_\varepsilon^{\mu\nu}$.

A question that arises naturally is whether the energy–momentum tensor constructed in (3.5) is determined uniquely. If we insist on the physical requirement that off the worldline this tensor should coincide with the original one (2.9), and if we enforce duality invariance, *a priori* the electromagnetic energy–momentum tensor is indeed determined only modulo a term, supported on the worldline, of the form,

$$\Delta T_{\rm em}^{\mu\nu}=(e^2+g^2)\int h^{\mu\nu}\delta^4(x-y)\,\mathrm{d}s,$$

where $h^{\mu\nu}$ is a symmetric tensor, of dimension 1 over length, constructed with u^μ, w^μ, $\mathrm{d}w^\mu/\mathrm{d}s$ etc. The question is now if there exists a tensor $h^{\mu\nu}$ for which the modified total energy–momentum tensor $\widehat{T}^{\mu\nu}\equiv T_{\rm em}^{\mu\nu}+\Delta T_{\rm em}^{\mu\nu}+m\int u^\mu u^\nu\delta^4(x-y)\mathrm{d}s$ is still conserved. Given the above form of $\Delta T_{\rm em}^{\mu\nu}$ the answer to this question is the same as in the case of charged particles, and it has been given in [2]: there exists no $h^{\mu\nu}\neq 0$ such that $\partial_\mu\widehat{T}^{\mu\nu}=0$, *whatever modified Lorentz–Dirac equation one imposes on the particle*. Given the above requirements, and the implicit assumptions that our dyon is point-like and spinless, the 4-momentum conservation fixes therefore (3.5) uniquely.

Our regularization appears also particularly useful for the derivation of effective equations of motion for extended objects, alternative to [6]. In the present case, e.g. the self-force can be obtained evaluating $F_\varepsilon^{\mu\nu}$ in (2.12) at the worldline $x=y(s)$, using (A.1), and then taking $\varepsilon\to 0$. The result is

$$H_\varepsilon^{\mu\nu}(y(s))=\frac{1}{8\pi\varepsilon}\,(u^\mu w^\nu-u^\nu w^\mu)-\frac{1}{6\pi}\left(u^\mu\frac{\mathrm{d}w^\nu}{\mathrm{d}s}-u^\nu\frac{\mathrm{d}w^\mu}{\mathrm{d}s}\right)+o(\varepsilon).$$

Using this in the regularized Lorentz equation for dyons, $\frac{\mathrm{d}p^\mu}{\mathrm{d}s}=\left(eF_\varepsilon^{\mu\nu}+g\widetilde{F}_\varepsilon^{\mu\nu}\right)u_\nu$, one sees that the divergent part renormalizes the mass, and that the finite part amounts to (2.8).

Our regularization scheme admits a simple interpretation in terms of the retarded Green function $G(x)=\frac{1}{2\pi}H(x^0)\delta(x^2)$ of the Laplacian $\Box=\partial_\mu\partial^\mu$. It is indeed immediately seen that the regularized potential (2.11) is produced by the regularized Green function $G_\varepsilon(x)=\frac{1}{2\pi}H(x^0)\delta(x^2-\varepsilon^2)$, where H is the Heaviside function, according to $A_\varepsilon^\mu(x)=\int\mathrm{d}^4z\,G_\varepsilon(x-z)\,j^\mu(z)$. This regularization extends naturally to arbitrary dimensions since in even space-times, $D=2n+4$, the Green function is $G=H(x^0)/2\pi^{n+1}(\mathrm{d}/\mathrm{d}x^2)^n\delta(x^2)$, while in odd ones, $D=2n+3$, it is $G=H(x^0)/2\pi^{n+1}(\mathrm{d}/\mathrm{d}x^2)^n[H(x^2)/\sqrt{x^2}]$, see [7]. The regularized Green function G_ε in arbitrary dimensions is then simply obtained operating in G the replacement $x^2\to x^2-\varepsilon^2$.

Interpreted in this way our method admits then a natural extension to extended objects in higher dimensions. For example, for an electric brane in D dimensions, minimally coupled to a p-form gauge field B, we can introduce a retarded regularized potential—in Lorentz gauge—according to $B_\varepsilon(x)=\int\mathrm{d}^Dz\,G_\varepsilon(x-z)\,j^{(p)}(z)$, where the current $j^{(p)}$ is the δ-function on the brane, i.e. its Poincarè dual. This potential gives rise to a field strength $(p+1)$-form $F_\varepsilon=\mathrm{d}B_\varepsilon$, that is regular on the brane, and hence to the regularized energy–momentum tensor $\Theta_\varepsilon^{\mu\nu}=\frac{1}{p!}\left[(F_\varepsilon F_\varepsilon)^{\mu\nu}-\frac{1}{2(p+1)}\eta^{\mu\nu}(F_\varepsilon F_\varepsilon)\right]$. Following the lines of the present paper it should then be possible to construct a finite and conserved energy–momentum tensor for a generic brane—taking its radiation reaction into account—providing thus a physical basis for the effective equations of motion postulated previously [6]. Moreover, the knowledge of this tensor should also allow a systematic analysis of the energy–momentum loss of an extended object, due to the emitted radiation. We hope to report on this construction soon.

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Appendix. Evaluation of $H_\varepsilon^{\nu\mu} j_{\varepsilon\mu}$ for $\varepsilon \to 0$

From (2.13) and (3.1) one obtains

$$H_\varepsilon^{\mu\nu} = \frac{1}{4\pi(uR)^3}[R^\mu u^\nu + (u^\rho w^\nu - u^\nu w^\rho)R_\rho R^\mu - (\mu \leftrightarrow \nu)]. \tag{A.1}$$

From this and (4.3) one sees that all terms appearing in the product $H_\varepsilon^{\nu\mu} j_{\varepsilon\mu}$ are schematically of the form

$$I_\varepsilon \equiv \varepsilon^2 \frac{R^{\mu_1}\cdots R^{\mu_N}}{(uR)^M} G(s_\varepsilon), \tag{A.2}$$

where G is a tensor constructed with u, w and b, all evaluated at $s_\varepsilon(x)$, whose tensorial structure we do not indicate explicitly. One has to apply this expression to a test function and to consider the limit $\varepsilon \to 0$. Proceeding as in (3.4) one obtains

$$\begin{aligned} I_\varepsilon(\varphi) &= \varepsilon^2 \int \mathrm{d}s \int \frac{\mathrm{d}^4x}{(ux)^{M-1}} 2\delta(x^2-\varepsilon^2)x^{\mu_1}\cdots x^{\mu_N} G(s)\varphi(x+y) \\ &= \frac{1}{\varepsilon^{M-N-5}} \int \mathrm{d}s \int \frac{\mathrm{d}^4x}{(ux)^{M-1}} 2\delta(x^2-1)x^{\mu_1}\cdots x^{\mu_N} G(s)[\varphi(y) + \varepsilon x^\alpha \partial_\alpha \varphi(y) + \cdots], \end{aligned} \tag{A.3}$$

where we have rescaled $x \to \varepsilon x$, and expanded $\varphi(\varepsilon x + y)$ in powers of ε. The values of M and N appearing in $H_\varepsilon^{\nu\mu} j_{\varepsilon\mu}$ are such that $M - N = 4, 5, 6, 7$. Therefore, as $\varepsilon \to 0$ at most the first three terms of the series above give a non-vanishing contribution. To conclude the evaluation of $I_\varepsilon(\varphi)$ one must eventually perform the integration over d^4x. This integration can be performed by taking multiple derivatives w.r.t. u^μ, considered as an independent variable, of the generating function,

$$\int \mathrm{d}^4x \frac{2\delta(x^2-1)}{(ux)^n} = \frac{\pi^{3/2}}{(u^2)^{n/2}} \frac{\Gamma\left(\frac{n}{2}-1\right)}{\Gamma\left(\frac{n+1}{2}\right)},$$

and setting eventually $u^2 = 1$. The x-integration gives thus rise to polynomials in u^μ.

One sees then that as $\varepsilon \to 0$ $I_\varepsilon(\varphi)$ reduces to a (finite) sum of terms of the kind $1/\varepsilon^l \int L(s)\partial\cdots\partial\varphi(y)\,\mathrm{d}s$, where $l = 0, 1, 2$ and at most two derivatives on φ appear. This means that I_ε, as ε goes to zero in the distributional sense, is supported on the worldline, becoming a sum of terms of the type $1/\varepsilon^l \int L(s)\partial\cdots\partial\delta^4(x-y)\,\mathrm{d}s$. More precisely, the terms in $H_\varepsilon^{\nu\mu} j_{\varepsilon\mu}$ with $M - N = 4$ converge to zero as $\varepsilon \to 0$, those with $M - N = 5, 6$ give rise to finite and simple pole contributions, while those with $M - N = 7$ give rise *a priori* also to double pole contributions. However, by direct inspection one sees that the double poles cancel. Indeed, from (A.1) and (4.3) one sees that the terms in $H_\varepsilon^{\nu\mu} j_{\varepsilon\mu}$ with $M - N = 7$ are given by $I_\varepsilon^7 = \frac{3\varepsilon^2}{(4\pi)^2(uR)^8} R_\mu(\eta^{\mu\nu} - u^\mu u^\nu)$. According to (A.2) and (A.3) we have ($M = 8, N = 1$),

$$I_\varepsilon^7(\varphi) = \frac{3}{(4\pi)^2\varepsilon^2} \int \mathrm{d}s \int \frac{\mathrm{d}^4x}{(ux)^7} 2\delta(x^2-1)x_\mu\,(\eta^{\mu\nu} - u^\mu u^\nu)\,[\varphi(y) + \varepsilon x^\alpha \partial_\alpha\varphi(y) + \cdots],$$

and the double pole cancels since $\int \frac{\mathrm{d}^4x}{(ux)^7} 2\delta(x^2-1)x_\mu = \frac{8\pi}{15} u_\mu$. Only finite and simple pole terms survive then in $H_\varepsilon^{\nu\mu} j_{\varepsilon\mu}$, and a straightforward but a bit lengthy calculation gives (4.7).

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